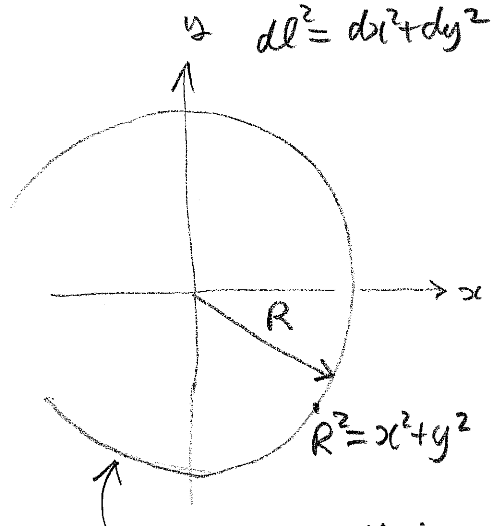
De Sitter Spacetime

Basic property: simplest admissible spacetime of constant curvature

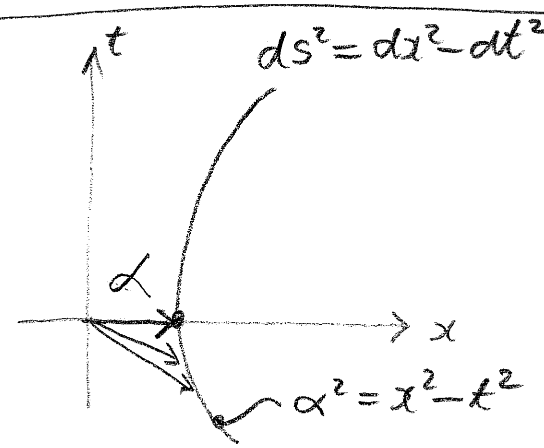
Project: Recover its properties from analogy with spheres in Euclidean space

2D Euclidean space



Circle = line, all of whose points are a constant distance R from the origin

1+1D Minkowski spacetime



Hyperbola = line, all of whose points are a constant interval $s = \alpha$ from origin

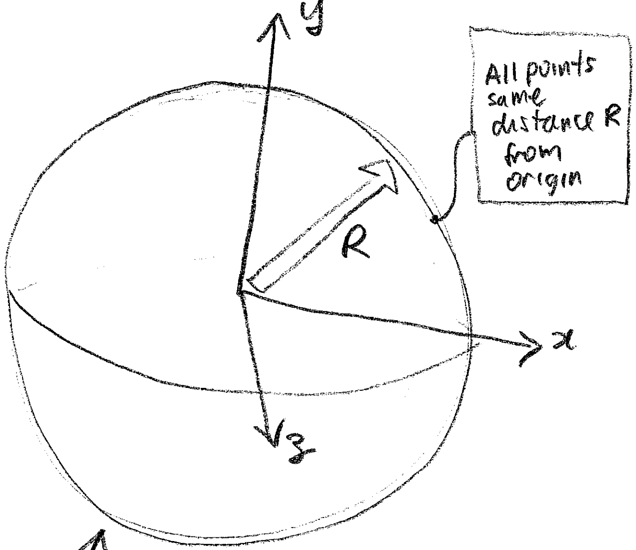
Typical point on hyperbola is (x, t)

$$(\text{Interval})^2 \text{ origin} \to (x,t)$$
$$= (x-0)^2 - (t-0)^2 = \alpha^2$$

Extend construction to spaces of higher dimension to generate surfaces of constant curvature

4-D Euclidean space

$dl^2 = dx^2 + dy^2 + dz^2 + du^2$



3-D dimensional hypersphere.

Surface is a 3-D space of constant curvature

$\boxed{x^2+y^2+z^2+u^2 = R^2}$

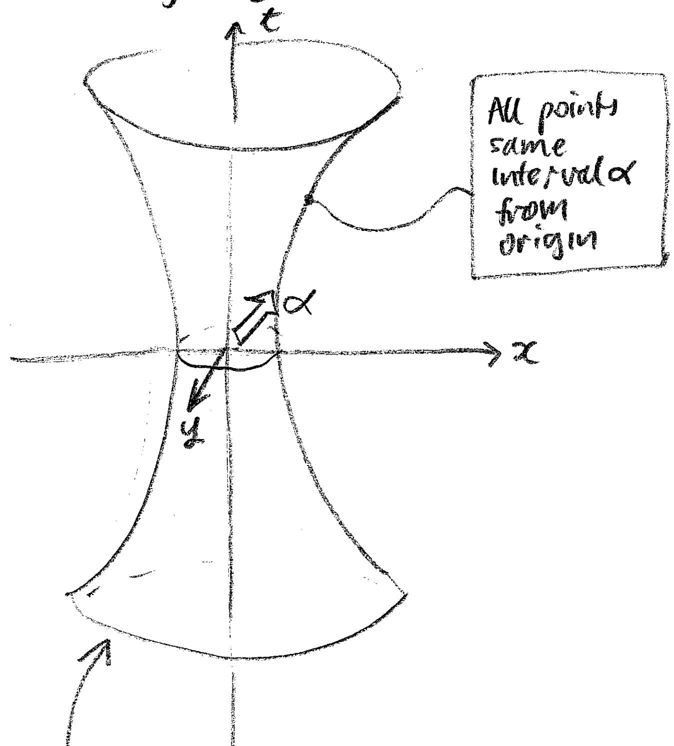
4d line element induces metrical structure on 3-D space

i.e. distance between infinitesimally close points on sphere

= distance between infinitesimally close points in 4D space

5-D Minkowski space

$ds^2 = dx^2 + dy^2 + dz^2 + du^2 - dt^2$



4-D hyperboloid made by rotating 2-D hyperbola into remaining dimensions about t-axis

Equation of hyperboloid

$\boxed{x^2+y^2+z^2+u^2 - t^2 = \alpha^2}$

4-D spacetime of constant curvature

metrical structure induced by 5-D metric.

This surface is the de Sitter spacetime

The imaginary coordinate trick

De Sitter spacetime is

$$ds^2 = dx^2 + dy^2 + dz^2 + du^2 - dt^2$$

restricted to

$$x^2 + y^2 + z^2 + u^2 - t^2 = \alpha^2$$

Make hyperboloid mimick hyperspere by introducing imaginary coordinate τ via

$$t = i\tau$$

De Sitter spacetime is

$$ds^2 = dx^2 + dy^2 + dz^2 + du^2 + d\tau^2$$

restricted to

$$x^2 + y^2 + z^2 + u^2 + \tau^2 = \alpha^2$$

Looks just like hypersphere !!
$\therefore$ Recycle results from analysis of sphere

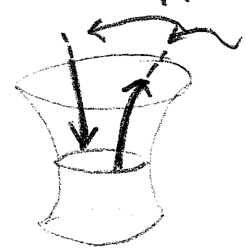
BEWARE! Two cases remain topologically distinct!

* sphere is S^4

can go equator → equator via North pole

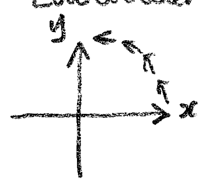
BUT

Hyperboloid is $S^3 \times R$



corresponding path does not connect

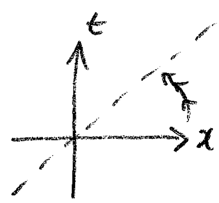
* Euclidean space:



Orthogonal transformation (rotation) smoothly rotates x axis into y axis

BUT

Minkowski spacetime

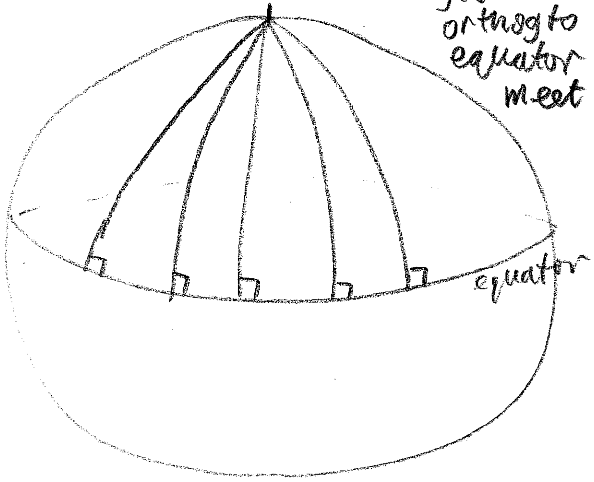


Orthogonal transformation (Lorentz transformation) CANNOT smoothly rotate x axis into t axis.
(only approaches $x=t$ in limit)

... but cautiously!

properties of geodesics

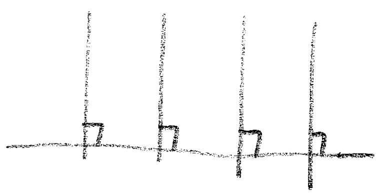
3-D spherical space



Interpretation:

In 3-D spherical space this construction does not give parallel lines

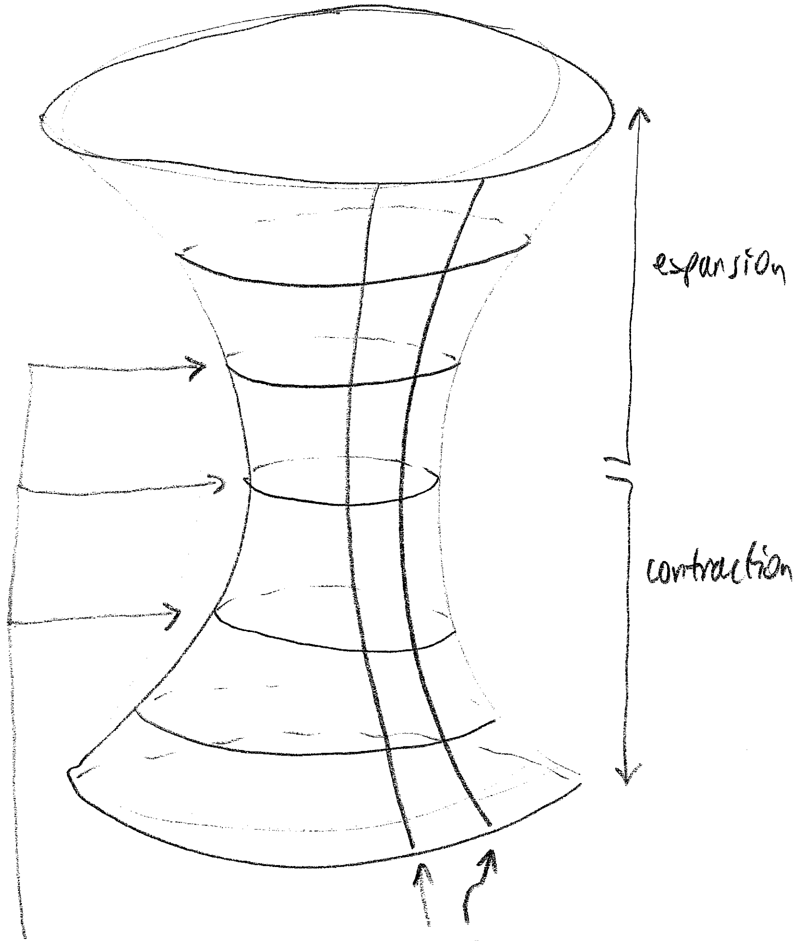
not possible



OR

Ants, starting at equator, all marching north eventually meet at north pole

de Sitter spacetime



space at different instants – contracts and then expands

geodesics = world lines of galaxies in free fall

Galaxies first rush together and then fly apart

De Sitter spacetime and the Einstein Field Equations

* De Sitter spacetime does <u>not</u> satisfy the (unaugmented) <u>source</u> <u>free</u> field equations of 1915.

<u>Quick Proof</u>: source free field equations ($\lambda = 0$)

$$\underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} = 0 \quad \left[\begin{array}{l}\text{contract}\\ \text{with}\\ g_{\mu\nu}\end{array} \right. \Longrightarrow \underbrace{R}_{\text{curvature scalar}} = 0$$

$$g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = R - \frac{1}{2} 4 R = -R$$

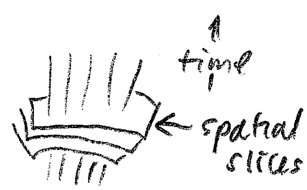
But $R = \text{constant} \neq 0$ for de Sitter spacetime

* De Sitter spacetime does solve source free field equations with $\lambda \neq 0$

<u>Plausibility</u>: $G_{\mu\nu} + \lambda g_{\mu\nu} = 0 \quad \left[\begin{array}{l}\text{contract}\\ \text{with}\\ g^{\mu\nu}\end{array} \right. \Longrightarrow -R + 4\lambda = 0$

$$R = 4\lambda = \text{constant} \neq 0$$

* De Sitter spacetime is RW spacetime i.e.



BUT

Different spatial slicing $\Longrightarrow$ geometry of space is spherical or Euclidean !

spacetime is static or not !

Simplest form of de-Sitter spacetime as Robertson Walker metric

Hawking + Ellis p. 124 +

$$ds^2 = dx^2 + dy^2 + dz^2 + du^2 - dt^2$$
restricted to
$$-t^2 + x^2 + y^2 + z^2 + u^2 = \alpha^2$$

transform to coordinates T, χ, θ, ϕ using

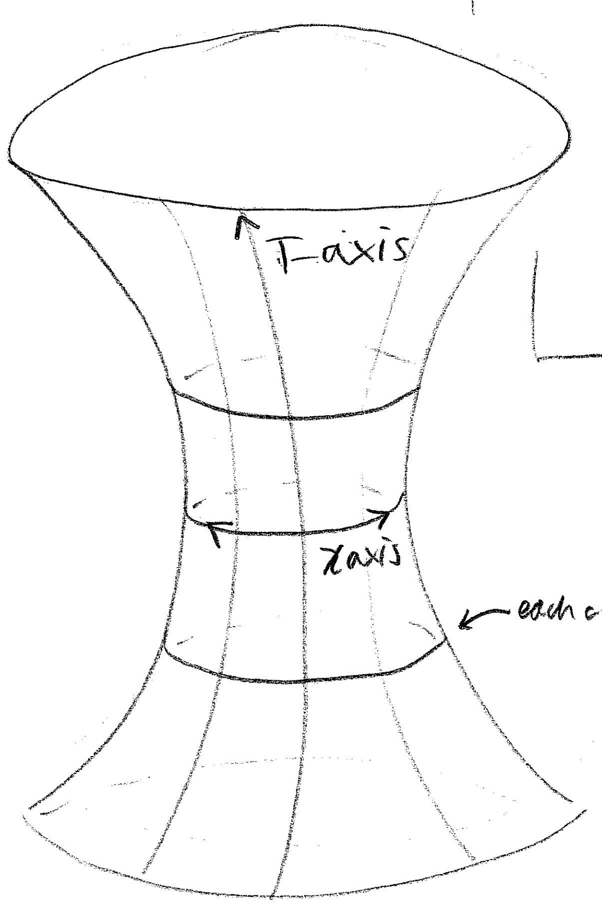
$$\alpha \sinh(T/\alpha) = t$$
$$\alpha \cosh(T/\alpha) \cos\chi = x$$
$$\alpha \cosh(T/\alpha) \sin\chi \cos\theta = y$$
$$\alpha \cosh(T/\alpha) \sin\chi \sin\theta \cos\phi = z$$
$$\alpha \cosh(T/\alpha) \sin\chi \sin\theta \sin\phi = u$$

$$ds^2 = -dT^2 + R^2(T)\, d\sigma^2$$

$$R = \alpha \cosh(T/\alpha)$$

$$d\sigma^2 = \left(d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

line element for 3-space with spherical geometry of constant curvature.



each circle = 3space of constant positive curvature

coordinate system covers ALL of hyperboloid

Metric satisfies dynamical equations of Robertson-Walker spacetime (due to Einstein field equations)
($\lambda \neq 0$, matter free, $k=1$)

$$\boxed{\frac{\ddot{R}}{R} = \frac{\Lambda}{3}}$$

satisfied since
$$R(T) = \alpha \cosh(T/\alpha)$$
$$\dot{R} = \sinh(T/\alpha)$$
$$\ddot{R} = \frac{1}{\alpha}\cosh(T/\alpha)$$

$$\therefore \frac{\ddot{R}}{R} = \frac{1}{\alpha^2} \quad \cdots \quad \boxed{\text{set } \frac{1}{\alpha^2} = \frac{\lambda}{3}}$$

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 = -\frac{k}{R^2} + \frac{\lambda}{3}}$$

satisfied since

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{1}{R^2} = \frac{1}{\alpha^2}\frac{\sinh^2(T/\alpha)}{\cosh^2(T/\alpha)} + \frac{1}{\alpha^2\cosh^2 T/\alpha}$$

$= \frac{k}{R^2}$ with $k=1$

$$= \frac{1}{\alpha^2}\underbrace{\left(\tanh^2\left(\frac{T}{\alpha}\right) + \operatorname{sech}^2\left(\frac{T}{\alpha}\right)\right)}_{=1 \text{ because my table of identities say so!}} = \frac{1}{\alpha^2} = \frac{\lambda}{3}$$

Form of de Sitter spacetime used in steady state cosmology

Hawking + Ellis

$$ds^2 = dx^2 + dy^2 + dz^2 + du^2 - dt^2$$
restricted to
$$-t^2 + x^2 + y^2 + z^2 + u^2 = \alpha^2$$

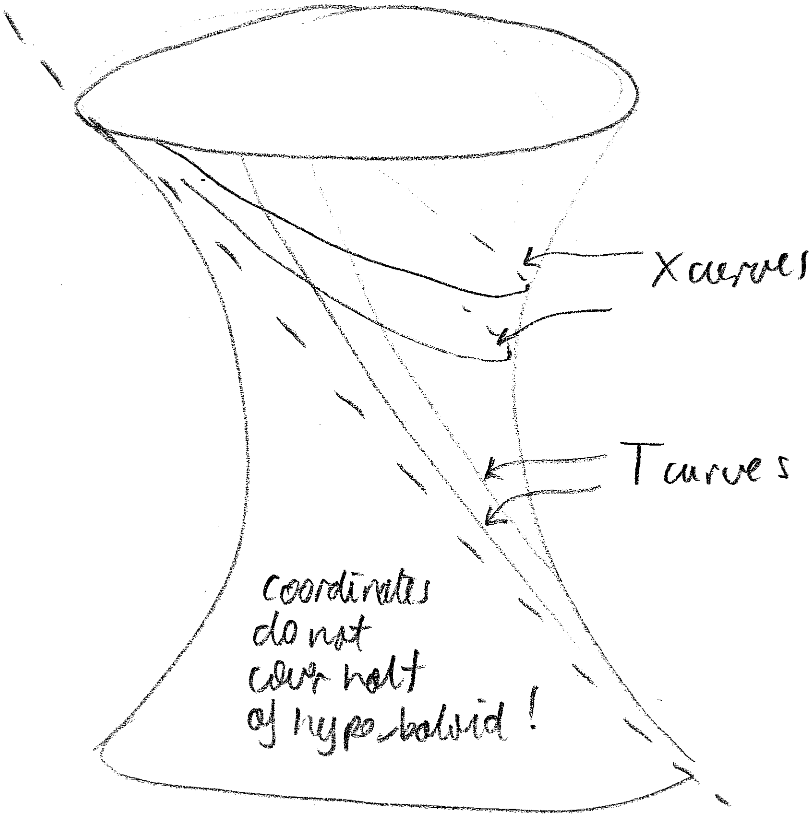
Transform to T, X, Y, Z using

$$T = \alpha \log \frac{x+t}{\alpha}$$

$$X = \frac{\alpha y}{x+t} \quad Y = \frac{\alpha z}{x+t} \quad Z = \frac{\alpha u}{x+t}$$

$$ds^2 = -dT^2 + \underbrace{\exp\left(\frac{2T}{\alpha}\right)}_{R(T)} \underbrace{(dX^2 + dY^2 + dZ^2)}_{\text{EUCLIDEAN!! line element}}$$

Represents space that is always expanding



i.e. Spacetime of steady state universe is extensible!

OOPS!!!

de Sitter metric of steady state cosmology satisfies dynamical equations of Robertson-Walker spacetimes

($\lambda \neq 0$, source free, $K=0$

$$\boxed{\frac{\ddot{R}}{R} = \frac{\lambda}{3}}$$

since

$$R(T) = \exp\left(\frac{2T}{\alpha}\right)$$

$$\ddot{R}(T) = \frac{4}{\alpha^2}\exp\left(\frac{2T}{\alpha}\right)$$

$$\frac{\ddot{R}}{R} = \frac{4}{\alpha^2} \longrightarrow \boxed{\text{set } \frac{4}{\alpha^2} = \frac{\lambda}{3}}$$

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 = \frac{\lambda}{3}}$$

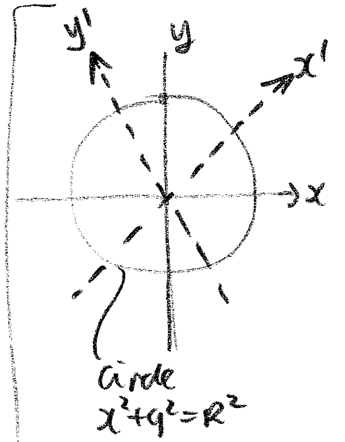
since

$$\frac{\dot{R}}{R} = \frac{2}{\alpha} \quad \therefore \left(\frac{\dot{R}}{R}\right)^2 = \frac{4}{\alpha^2}$$

A piece of the de Sitter spacetime can also look like a static spacetime

After Weyl, space-time-matter

Discover via analogy with sphere:



circle is preserved by rotation of axes

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

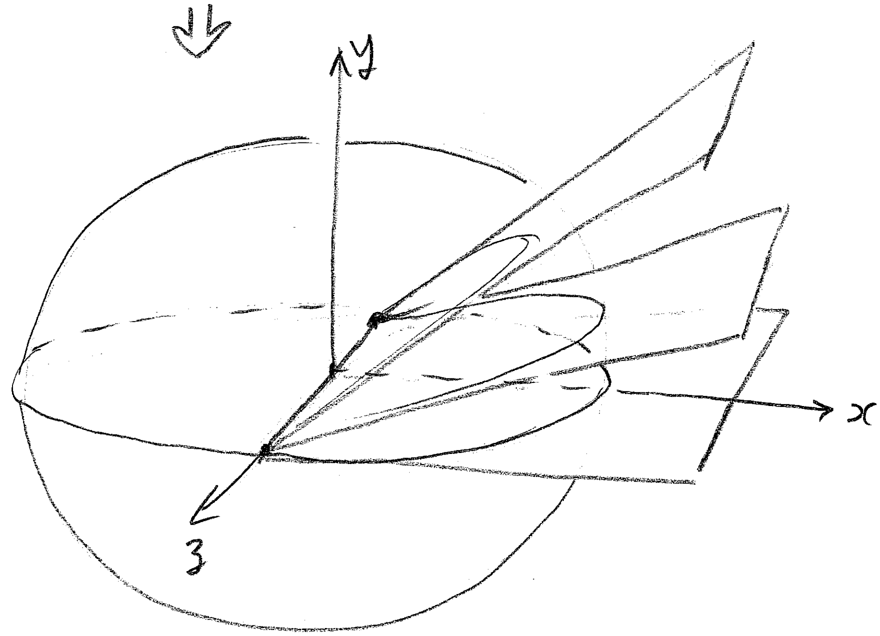
since

$$x'^2 = x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta$$
$$y'^2 = x^2\sin^2\theta - 2xy\cos\theta\sin\theta + y^2\cos^2\theta$$

so that

$$x'^2+y'^2 = x^2+y^2$$

For case of sphere



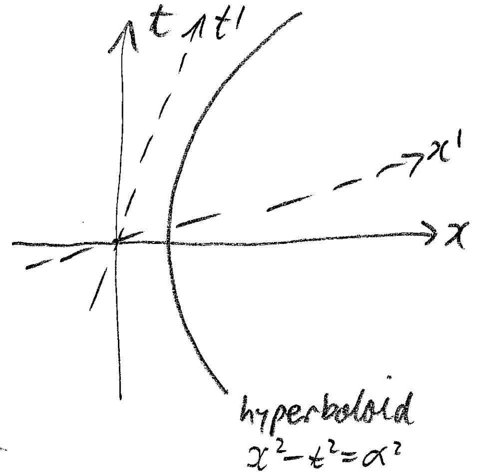
Slice circle with different flat planes, each rotated about z-axis

$\Downarrow$

sphere intersects plane in same circle

Analogous construction for de Sitter hyperboloid:

1+1 dim minkowski spacetime



Hyperboloid is preserved by Lorentz transformation

$$t' = t\cosh\theta + x\sinh\theta$$
$$x' = t\sinh\theta + x\cosh\theta$$

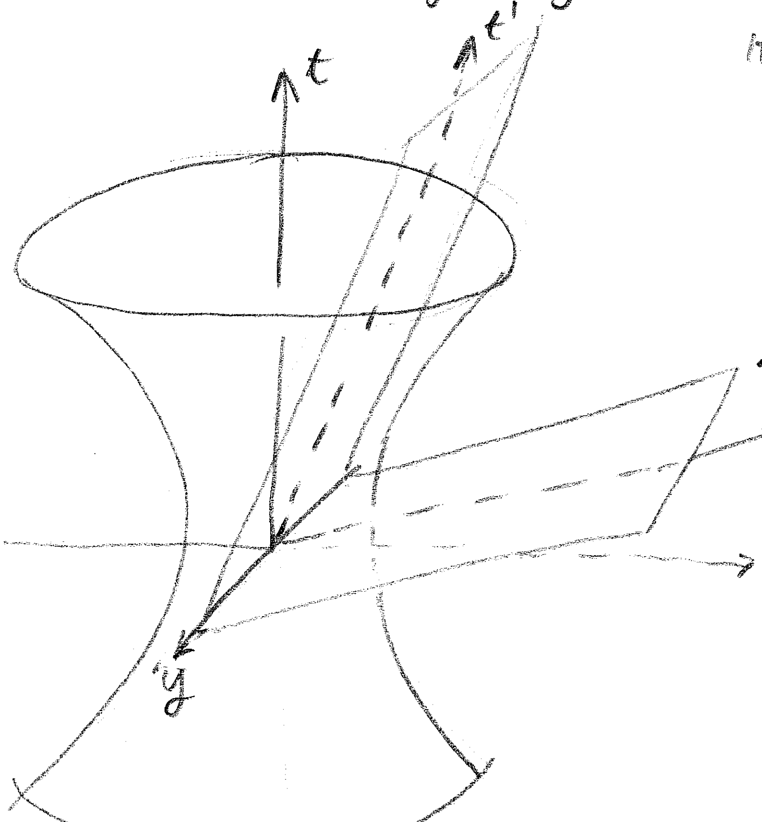
since by direct substitution*

$$x'^2 - t'^2 = x^2 - t^2$$

$\therefore$ Full de Sitter hyperboloid $x^2+y^2+z^2+u^2-t^2=\alpha^2$ is left unchanged by above Lorentz transformation

it becomes

$$x'^2+y'^2+z'^2+u'^2-t'^2=\alpha^2$$



$\therefore$ 3 space intersecting this x' plane is the same for all values of θ in above Lorentz transformation

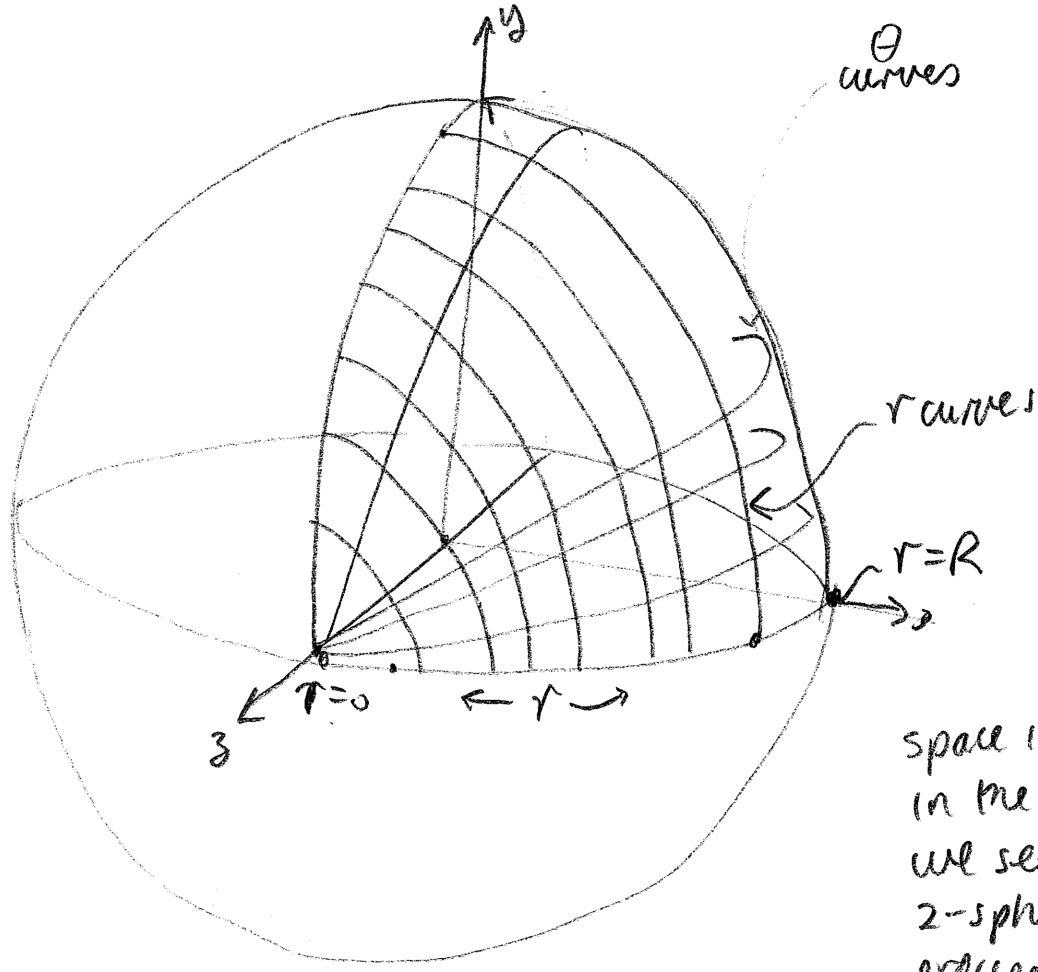
$$* \quad x'^2 = x^2\cosh^2\theta + \cancel{2xt\cosh\theta\sinh\theta} + t^2\sinh^2\theta$$
$$t'^2 = x^2\sinh^2\theta + \cancel{2xt\cosh\theta\sinh\theta} + t^2\cosh^2\theta$$
$$x'^2 - t'^2 = (x^2-t^2)\underbrace{(\cosh^2\theta - \sinh^2\theta)}_{1} = x^2 - t^2$$

spherical space is
$dl^2 = dx^2 + dy^2 + dz^2 + du^2$
restricted to
$x^2 + y^2 + z^2 + u^2 = R^2$

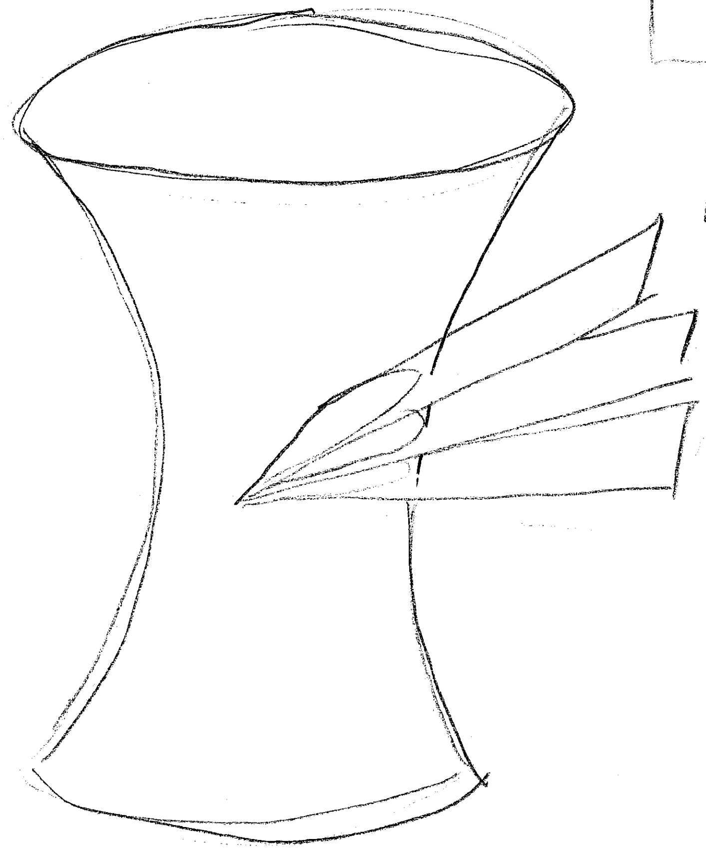
transform to coordinates (r, θ, z, u) where

$x = r\cos\theta$
$y = r\sin\theta$

$$\boxed{\begin{array}{c} dl^2 = dr^2 + r^2 d\theta^2 + dz^2 + du^2 \\ \text{restricted to } r^2 + z^2 + u^2 = R^2 \end{array}}$$



space is static in the sense that we see the same 2-sphere as we proceed along r-curves make the spacetime look static

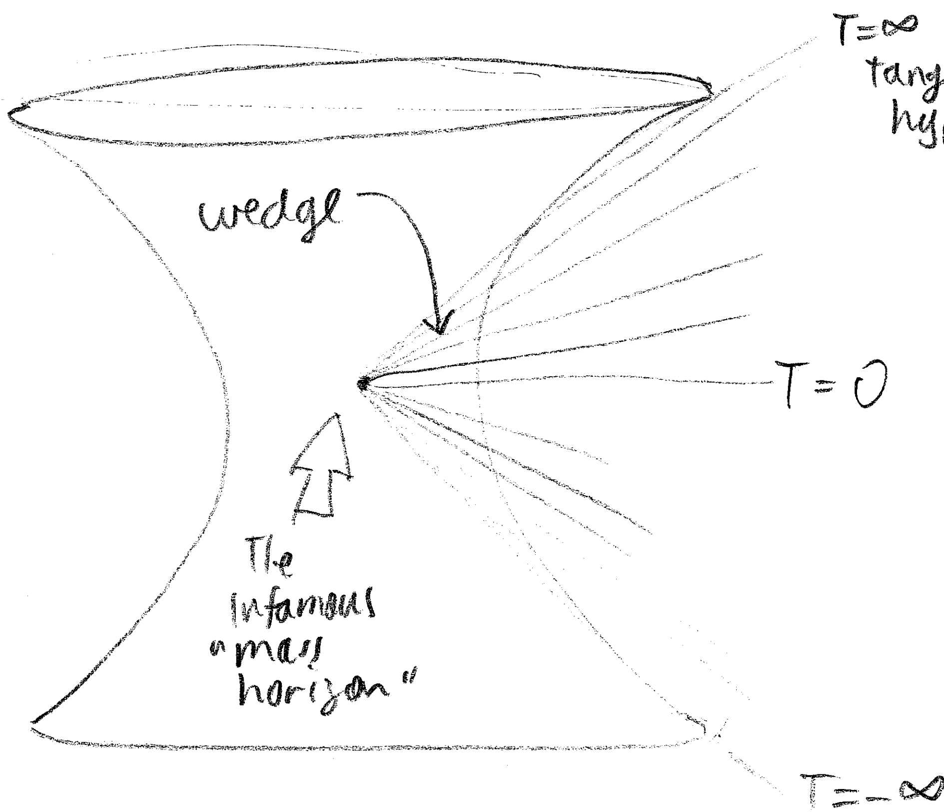


many different x' planes parameterized by θ of Lorentz transformation

use this θ as a new coordinate in timelike direction. Relabel it as

$\theta = T$

New coordinates will only cover a wedge of the hyperboloid



De Sitter spacetime is
$$ds^2 = dx^2+dy^2+dz^2+du^2-dt^2$$
restricted to
$$x^2+y^2+z^2+u^2-t^2=\alpha^2$$

Transform to (T, r, y, z, u) using
$$t = r\sinh T$$
$$x = r\cosh T$$

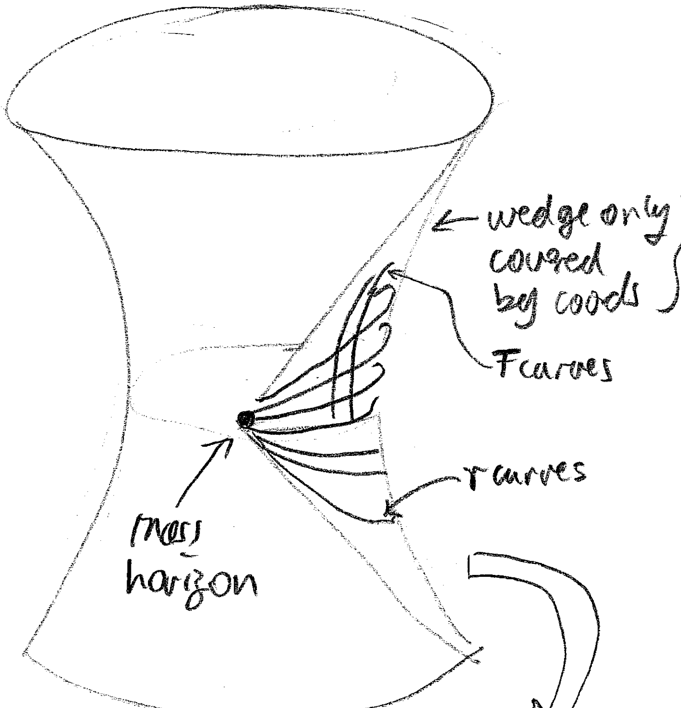
$$x^2-t^2 = r^2\cosh^2 T - r^2\sinh T = r^2$$

$$dt = dr\sinh T + dT\, r\cosh T$$
$$dx = dr\cosh T + dT\, r\sinh T$$
$$\therefore dx^2-dt^2 = dr^2 - r^2 dT^2$$

$$ds^2 = dr^2+dy^2+dz^2+du^2-r^2dT^2$$
restricted to
$$r^2+y^2+z^2+u^2=\alpha^2$$

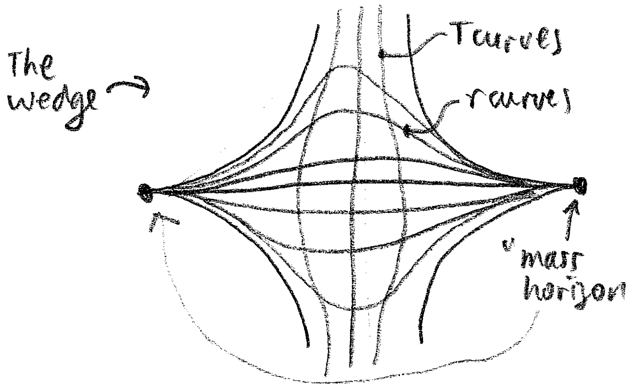
By construction. for each value of T, we see the same space

= (spherical space constant curvature: Radius curvature $=\alpha$)



only see $\frac{1}{2}$ of space

i.e. If we think of T as our time as we, as observers, move along the T curves of the hyperboloid, we see the same space of constant curvature α



Go through exercise of restricting line element to hyperboloid

$$ds^2 = dr^2 + dy^2 + dz^2 + du^2 - r^2 dT^2$$

restricted to $r^2 + y^2 + z^2 + u^2 = \alpha^2$

Relabel $r \to x_1 \quad z \to x_3$
$y \to x_2 \quad u \to x_4$

$$ds^2 = \sum_{i=1}^{4} dx_i^2 - x_1^2 dT^2$$

restricted to $\sum_i x_i^2 = \alpha^2$

i.e. $\sum_i x_i dx_i = 0$

Use this to elimate $x_4 = u$ from ds^2

$$dx_4 = du = \frac{x_1 dx_1 + x_2 dx_2 + x_3 dx_3}{x_4^2}$$

$$\therefore dx_4^2 = \underbrace{\sum_{i=1}^{3} (x_i dx_i)^2}_{\sum_{i=1}^{3} \sum_{j=1}^{3} x_i x_j dx_i dx_j} / (\alpha^2 - x_1^2 - x_2^2 - x_3^2)$$

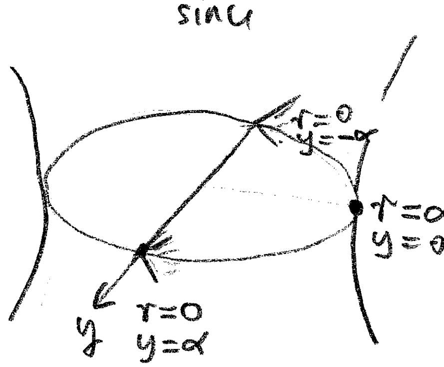
Substitute into ds^2 to recover

$$ds^2 = \underbrace{\sum_{i,j=1}^{3} \left[\delta_{ij} + \frac{x_i x_j}{(\alpha^2 - x_1^2 - x_2^2 - x_3^2)} \right] dx_i dx_j}_{\text{line element for 3D space constant positive curvature. (Expression of Einstein 1917)}} - x_1^2 dT^2$$

$r = x_1$

$$\boxed{ds^2 = \underbrace{r^2}_{0 \le r \le \alpha \text{ since}} \underbrace{dT^2}_{-\infty < T < \infty} - d\omega(\alpha)^2}$$

— spacetime static in sense that space of all constant T slices is the same (no expansion or contraction)



A moral?

Algebraic methods: Analyse de Sitter spacetime as algebraic problem in manipulation of variables T, r, y, z, u.

$$ds^2 = r^2 dT - d\sigma^2$$

At $r=0$, coordinate time interval dT

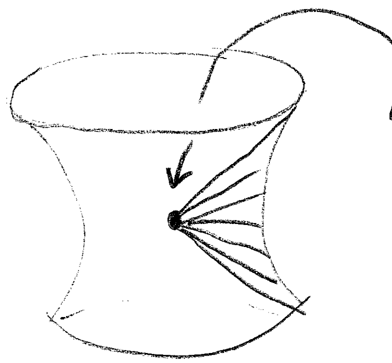
$\Updownarrow$

No proper time elapsed $ds^2 = 0$

Is this singular behavior of the spacetime?

versus

Geometric methods: Drive analysis by geometric pictures and analogies



Odd behavior at $r=0$, since this is the point at which all the r-curves cross

But no singularity in spacetime. de Sitter spacetime is <u>homogeneous</u>.

$\therefore$ One point singular $\Rightarrow$ All points singular !!!

BUT Geometric methods require algebraic methods to supply results onto which geometric pictures are pasted.

Geometric methods:
Also notice that static coords, coords of steady state U do not cover all of de Sitter spacetime.
i.e. <u>more</u> spacetime beyond coordinates $= \pm\infty$!